

TOEPLITZ AND ASYMPTOTIC TOEPLITZ OPERATORS ON $H^2(\mathbb{D}^n)$

AMIT MAJI, JAYDEB SARKAR, AND SRIJAN SARKAR

ABSTRACT. Let M_{z_j} denote the multiplication operator on the Hardy space $H^2(\mathbb{D}^n)$ (over the unit polydisc \mathbb{D}^n) by the j^{th} coordinate function z_j , $j = 1, \dots, n$, and let T be a bounded linear operator on $H^2(\mathbb{D}^n)$. Then:

(i) T is a Toeplitz operator (that is, $T = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}$ for some $\varphi \in L^\infty(\mathbb{T}^n)$) if and only if $M_{z_j}^* T M_{z_j} = T$ for all $j = 1, \dots, n$.

(ii) T is an asymptotic Toeplitz operator if and only if $T = \text{Toeplitz} + \text{compact}$.

The case $n = 1$ is the well known results of Brown and Halmos, and Feintuch, respectively. Along the way we generalize some of the recent results of Chalendar and Ross to vector-valued Hardy spaces.

1. INTRODUCTION

Although concrete bounded linear operators on Hilbert spaces exist in great variety and can exhibit interesting properties, one of the main concerns of function theory and operator theory has generally been the study of operators which are connected with the spaces of holomorphic and integrable functions. The class of Toeplitz and analytic Toeplitz operators have turned out to be one of the most important classes of concrete operators from this point of view.

Toeplitz operators on the Hardy space (or, on the l^2 space) were first studied by O. Toeplitz (and then by P. Hartman and A. Wintner in [15]). However, a systematic study of Toeplitz operators was triggered by the seminal paper of Brown and Halmos [4] on algebraic properties of Toeplitz operators on $H^2(\mathbb{D})$ (the Hardy space over the open unit disc \mathbb{D} in \mathbb{C}). The study of Toeplitz operators on Hilbert spaces of holomorphic functions (like the Hardy space, the Bergman space and the weighted Bergman spaces) on domains in \mathbb{C}^n is also one of the very active area of current research that brings together several areas of mathematics. For more information on this direction of research, we refer the reader to [3], [6], [8], [9], [11], [20] and the references therein.

Recall that the well-known Brown-Halmos theorem characterizes Toeplitz operators on $H^2(\mathbb{D})$ as follows (see the matricial characterization, Theorem 6 in [4]): Let T be a bounded linear operator on $H^2(\mathbb{D})$. Then T is a Toeplitz operator if and only if

$$M_z^* T M_z = T.$$

2010 *Mathematics Subject Classification.* 47B35, 47A13, 47A45, 30H10, 30H50, 47A20, 47B07, 15B05.

Key words and phrases. Toeplitz operators, Hardy space over the polydisc, vector-valued Hardy spaces, compact operators, quotient spaces, model spaces.

One of the main results of this paper is the following generalization (see Theorem 4.1) of Brown-Halmos theorem: A bounded linear operator T on $H^2(\mathbb{D}^n)$ is a Toeplitz operator if and only if

$$M_{z_j}^* T M_{z_j} = T,$$

for all $j = 1, \dots, n$ (see Section 2 for notation and background definitions).

The notion of Toeplitzness was extended to more general settings by Barriá and Halmos [2], Feintuch [10], and Chalendar and Ross [5] (see also Popescu [12] for Toeplitzness in the non-commutative setting). Accordingly, following Feintuch (and Barriá and Halmos [2], and Chalendar and Ross) we shall say that a bounded linear operator T on $H^2(\mathbb{D})$ is (uniformly) *asymptotically Toeplitz* if $\{M_z^{*m} T M_z^m\}_{m \geq 1}$ converges in operator norm.

The following theorem due to Feintuch [10] gives a remarkable characterization of asymptotically Toeplitz operators: A bounded linear operator T on $H^2(\mathbb{D})$ is asymptotically Toeplitz if and only if $T = \text{Toeplitz} + \text{compact}$.

In Theorem 4.4, we prove the following generalization of Feintuch's theorem: A bounded linear operator T on $H^2(\mathbb{D}^n)$ is asymptotically Toeplitz (see Definition 4.3) if and only if $T = \text{Toeplitz} + \text{compact}$.

On the other hand, let \mathcal{E} be a Hilbert space and $\Theta \in H_{\mathcal{B}(\mathcal{E})}^\infty(\mathbb{D})$ be an inner multiplier [17]. Then the *model operator* S_Θ [11] corresponding to Θ is the compression of M_z to the *model space*

$$\mathcal{K}_\Theta := H_{\mathcal{E}}^2(\mathbb{D}) \ominus \Theta H_{\mathcal{E}}^2(\mathbb{D}),$$

where $H_{\mathcal{E}}^2(\mathbb{D})$ denotes the Hardy space of \mathcal{E} -valued functions. Therefore

$$S_\Theta = P_{\mathcal{K}_\Theta} M_z|_{\mathcal{K}_\Theta},$$

where $P_{\mathcal{K}_\Theta}$ denotes the orthogonal projection from $H_{\mathcal{E}}^2(\mathbb{D})$ onto \mathcal{K}_Θ . Note that $\mathcal{K}_\Theta^\perp = \Theta H_{\mathcal{E}}^2(\mathbb{D})$ is an M_z -invariant subspace of $H_{\mathcal{E}}^2(\mathbb{D})$ and $S_\Theta^* = M_z^*|_{\mathcal{K}_\Theta} \in \mathcal{B}(\mathcal{K}_\Theta)$.

It is well known that if T on a Hilbert space \mathcal{H} is a contraction and $T^{*m} \rightarrow 0$ in strong operator topology, then there exist a Hilbert space \mathcal{E} and an inner multiplier $\Theta \in H_{\mathcal{B}(\mathcal{E})}^\infty(\mathbb{D})$ such that T and S_Θ are unitarily equivalent. In other words, model operators represent a large class of bounded linear operators on Hilbert spaces (see Sz.-Nagy and Foias [17]). In the case $\mathcal{E} = \mathbb{C}$, S_Θ is called a *Jordan block* [17].

A basic question is now to characterize those $T \in \mathcal{B}(\mathcal{K}_\Theta)$ for which

$$S_\Theta^* T S_\Theta = T.$$

Similarly, characterize those $T \in \mathcal{B}(\mathcal{K}_\Theta)$ for which

$$S_\Theta^{*m} T S_\Theta^m \rightarrow A,$$

in norm, for some $A \in \mathcal{B}(\mathcal{K}_\Theta)$.

These questions have been answered by Chalendar and Ross [5] in the case where S_Θ is a Jordan block: Let S_Θ on $\mathcal{K}_\Theta(\subseteq H^2(\mathbb{D}))$ be a Jordan block and $T \in \mathcal{B}(\mathcal{K}_\Theta)$. Then (i) $S_\Theta^* T S_\Theta = T$ if and only if $T = 0$, and (ii) $\{S_\Theta^{*m} T S_\Theta^m\}_{m \geq 1}$ converges in norm if and only if T is compact.

In this paper we extend the above results in different settings, such as vector-valued Hardy space and Hardy space over polydisc.

The remainder of the paper is organized as follows. In Section 2 (preliminary section), we set up notations, definitions and results needed further. In Section 3, we give general results concerning asymptotic Toeplitz operators and compact operators on model spaces. Section 4 deals with the Hardy space over \mathbb{D}^n . In Section 5, we generalize some results obtained for the vector-valued Hardy space in Section 3 in the setting of $H^2(\mathbb{D}^n)$.

2. PRELIMINARIES

Let $n \geq 1$ and \mathbb{D}^n be the open unit polydisc in \mathbb{C}^n . In the sequel, \mathbf{z} will always denote a vector $\mathbf{z} = (z_1, \dots, z_n)$ in \mathbb{C}^n .

The *Hardy space* $H^2(\mathbb{D}^n)$ over \mathbb{D}^n is the Hilbert space of all holomorphic functions f on \mathbb{D}^n such that

$$\|f\|_{H^2(\mathbb{D}^n)} := \left(\sup_{0 \leq r < 1} \int_{\mathbb{T}^n} |f(re^{i\theta_1}, \dots, re^{i\theta_n})|^2 d\boldsymbol{\theta} \right)^{\frac{1}{2}} < \infty,$$

where $d\boldsymbol{\theta}$ is the normalized Lebesgue measure on the torus \mathbb{T}^n , the distinguished boundary of \mathbb{D}^n . Let $(M_{z_1}, \dots, M_{z_n})$ denote the n -tuple of multiplication operators by the coordinate functions $\{z_i\}_{i=1}^n$, that is,

$$(M_{z_i}f)(\mathbf{w}) = w_i f(\mathbf{w}) \quad (\mathbf{w} \in \mathbb{D}^n, i = 1, \dots, n).$$

We will often identify $H^2(\mathbb{D}^n)$ with the n -fold Hilbert space tensor product of one variable Hardy space as $H^2(\mathbb{D}) \otimes \dots \otimes H^2(\mathbb{D})$. In this identification, M_{z_i} can be represented as

$$I_{H^2(\mathbb{D})} \otimes \dots \otimes \underbrace{M_z}_{i^{th} \text{ place}} \otimes \dots \otimes I_{H^2(\mathbb{D})} \quad (i = 1, \dots, n).$$

Also one can identify the Hardy space (via the radial limits of functions in $H^2(\mathbb{D}^n)$) with the closed subspace of $L^2(\mathbb{T}^n)$ in the following sense: Let $\{e_{\mathbf{s}} : \mathbf{s} \in \mathbb{Z}^n\}$ be the orthonormal basis of $L^2(\mathbb{T}^n)$, where $\mathbf{s} = (s_1, \dots, s_n) \in \mathbb{Z}^n$ and $e_{\mathbf{s}} = e^{i\theta_1 s_1} \dots e^{i\theta_n s_n}$. Then a function

$$f = \sum_{\mathbf{s} \in \mathbb{Z}^n} a_{\mathbf{s}} e_{\mathbf{s}} \in L^2(\mathbb{T}^n),$$

is the radial limit function of some f in $H^2(\mathbb{D}^n)$ if and only if $a_{\mathbf{s}} = 0$ whenever at least one of the s_j , $j = 1, \dots, n$, is negative. In particular, the set of all monomials $\{\mathbf{z}^{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^n\}$ form an orthonormal basis for $H^2(\mathbb{D}^n)$, where $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ and $\mathbf{z}^{\mathbf{k}} = z_1^{k_1} \dots z_n^{k_n}$ (cf. [1], [19]). We use $P_{H^2(\mathbb{D}^n)}$ to denote the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$, that is,

$$P_{H^2(\mathbb{D}^n)} \left(\sum_{\mathbf{s} \in \mathbb{Z}^n} a_{\mathbf{s}} e_{\mathbf{s}} \right) = \sum_{\mathbf{s} \in \mathbb{N}^n} a_{\mathbf{s}} e_{\mathbf{s}},$$

for all $\sum_{\mathbf{s} \in \mathbb{Z}^n} a_{\mathbf{s}} e_{\mathbf{s}}$ in $L^2(\mathbb{T}^n)$.

For $\varphi \in L^\infty(\mathbb{T}^n)$, the *Toeplitz operator* with symbol φ is the operator $T_\varphi \in \mathcal{B}(H^2(\mathbb{D}^n))$ defined by

$$T_\varphi f = P_{H^2(\mathbb{D}^n)}(M_\varphi f) = P_{H^2(\mathbb{D}^n)}(\varphi f) \quad (f \in H^2(\mathbb{D}^n)),$$

where M_φ is the Laurant operator on $L^2(\mathbb{T}^n)$ (that is, $M_\varphi g = \varphi g$ for all $g \in L^2(\mathbb{T}^n)$). In other words,

$$T_\varphi = P_{H^2(\mathbb{D}^n)} M_\varphi|_{H^2(\mathbb{D}^n)}.$$

For the relevant results on Toeplitz operators on $H^2(\mathbb{D}^n)$ we refer the reader to [3, 6, 8, 16, 18] and references therein.

3. ASYMPTOTIC TOEPLITZ OPERATORS ON $H_{\mathcal{E}}^2(\mathbb{D})$

The main purpose of this section is to characterize the compact operators on the model space $H_{\mathbb{C}^p}^2(\mathbb{D})/\Theta H_{\mathbb{C}^p}^2(\mathbb{D})$, where $\Theta \in H_{\mathcal{B}(\mathbb{C}^p)}^\infty(\mathbb{D})$ is an inner function. We note that this result for Jordan block case (when $p = 1$) can be found in Chalendar and Ross [5]. Moreover, our proof seems more shorter (for instance, compare Proposition 3.4 with Proposition 2.10 in [5]).

We start with an useful lemma of independent interest which generalizes Lemma 2.1 of [5]. However, the key ideas of the proof is due to Chalendar and Ross [5].

Lemma 3.1. *Let \mathcal{H} be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ be a compact operator. If R and T are contractions on \mathcal{H} , and if $R^{*m} \rightarrow 0$ in strong operator topology, then $R^{*m}AT \rightarrow 0$ in norm.*

Proof. We first observe that $R^{*m} \rightarrow 0$, as $m \rightarrow \infty$, uniformly on compact subsets of \mathcal{H} in strong operator topology. Indeed, let E be a compact subset of \mathcal{H} . Now, the hypothesis implies that for each $\epsilon > 0$ and $h \in E$ there exists a positive integer m_0 (depends on both h and ϵ) such that

$$\|R^{*m}h\| < \epsilon \quad (m \geq m_0).$$

Since R^{*m_0} is continuous, there exists $r_{h,\epsilon} > 0$ such that

$$\|R^{*m_0}f\| < \epsilon \quad (f \in B_{r_{h,\epsilon}}(h)),$$

where $B_{r_{h,\epsilon}}(h) = \{f \in \mathcal{H} : \|f - h\| < r_{h,\epsilon}\}$. Furthermore, since $\|R^*\| \leq 1$, we have that

$$\|R^{*m}f\| < \epsilon \quad (m > m_0, f \in B_{r_{h,\epsilon}}(h)).$$

Using compactness of E it follows that

$$E \subset \bigcup_{i=1}^N B_{r_{h_i,\epsilon}}(h_i),$$

for some $\{h_1, \dots, h_N\} \subseteq E$. Now setting $M = 1 + \max\{m_{h_i,\epsilon} : 1 \leq i \leq N\}$ we have that

$$\|R^{*m}g\| < \epsilon,$$

for all $m \geq M$ and $g \in E$. This says that $R^{*m} \rightarrow 0$ uniformly on E in strong operator topology.

Note finally that $E := \overline{A(B_1(0))}$ is compact and $T\overline{B_1(0)} \subseteq \overline{B_1(0)}$. Then combined with the above observation this implies that for $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that

$$\|R^{*m}AT\| = \sup_{x \in \overline{B_1(0)}} \|R^{*m}ATx\| \leq \sup_{y \in \overline{B_1(0)}} \|R^{*m}Ay\| \leq \sup_{h \in E} \|R^{*m}h\| < \epsilon,$$

for all $m \geq M$. This completes the proof. \square

Here let us observe, before we proceed further, that adapting similar concepts and techniques used by Brown and Halmos in [4], one can prove the following characterization of Toeplitz operators on a vector-valued Hardy space.

Theorem 3.2. *Let \mathcal{E} be a Hilbert space and $T \in \mathcal{B}(H_{\mathcal{E}}^2(\mathbb{D}))$. Then T is a Toeplitz operator if and only if $M_z^* T M_z = T$.*

In the theorem below, we generalize the Feintuch's characterization [10] (see also Theorem F, page 195, [18]) of asymptotic Toeplitz operators on Hardy space to asymptotic Toeplitz operators on Hardy space of finite multiplicity.

Theorem 3.3. *Let $T, A \in \mathcal{B}(H_{\mathbb{C}^p}^2(\mathbb{D}))$ and $M_z^{*m} T M_z^m \rightarrow A$ in norm. Then A is a Toeplitz operator and $(T - A)$ is compact. Conversely, if A is a Toeplitz operator and $T - A$ is a compact operator, then T is asymptotically Toeplitz.*

Proof. It follows that

$$\|M_z^{*(m+1)} T M_z^{m+1} - M_z^* A M_z\| \leq \|M_z^{*m} T M_z^m - A\| \rightarrow 0$$

as $m \rightarrow \infty$. This and the triangle inequality yields $A = M_z^* A M_z$. Now let $R_m = M_z^m M_z^{*m}$ and

$$Q_m = I - R_m.$$

Further, let $P_{\mathbb{C}^p}$ denote the orthogonal projection of $H_{\mathbb{C}^p}^2(\mathbb{D})$ onto the space of (\mathbb{C}^p -valued) constant functions. Since $M_z M_z^* = I_{H_{\mathbb{C}^p}^2(\mathbb{D})} - P_{\mathbb{C}^p}$, it follows that

$$Q_m = \sum_{k=0}^{m-1} M_z^k P_{\mathbb{C}^p} M_z^{*k} \quad (m \geq 1).$$

Then Q_m , $m \geq 1$, is a finite rank operator, and therefore

$$F_m = (T - A)Q_m + Q_m(T - A) - Q_m(T - A)Q_m \quad (m \geq 1),$$

is also a finite rank operator. Moreover

$$(T - A) - F_m = R_m(T - A)R_m \quad (m \geq 1),$$

yields

$$\|(T - A) - F_m\| = \|R_m(T - A)R_m\| \leq \|M_z^{*m} T M_z^m - A\| \rightarrow 0,$$

as $m \rightarrow \infty$. So $T - A$ is compact as desired.

The converse follows from Lemma 3.1. This completes the proof. \square

We have the following result in the model space setting.

Proposition 3.4. *Let $\Theta \in H_{\mathcal{B}(\mathcal{E})}^\infty(\mathbb{D})$ be an inner multiplier and $T \in \mathcal{B}(\mathcal{K}_\Theta)$. Assume that $\Theta(e^{i\theta})$ is invertible a.e. Then $S_\Theta^* T S_\Theta = T$ if and only if $T = 0$.*

Proof. Notice that $\mathcal{K}_\Theta^\perp = \Theta H_{\mathcal{E}}^2(\mathbb{D})$ is an M_z -invariant subspace of $H_{\mathcal{E}}^2(\mathbb{D})$. Then

$$T P_{\mathcal{K}_\Theta} = (S_\Theta^* T S_\Theta) P_{\mathcal{K}_\Theta} = (P_{\mathcal{K}_\Theta} M_z^*|_{\mathcal{K}_\Theta} T P_{\mathcal{K}_\Theta} M_z|_{\mathcal{K}_\Theta}) P_{\mathcal{K}_\Theta} = M_z^* T P_{\mathcal{K}_\Theta} M_z P_{\mathcal{K}_\Theta} = M_z^* T P_{\mathcal{K}_\Theta} M_z,$$

that is,

$$T P_{\mathcal{K}_\Theta} = M_z^* (T P_{\mathcal{K}_\Theta}) M_z,$$

and therefore $TP_{\mathcal{K}_\Theta}$ is a Toeplitz operator on $H_\mathcal{E}^2(\mathbb{D})$. Consequently, there exists $\Psi \in L_{\mathcal{B}(\mathcal{E})}^\infty(\mathbb{T})$ [17] such that $TP_{\mathcal{K}_\Theta} = T_\Psi$. Since M_Θ is an analytic Toeplitz operator, it follows that

$$0 = TP_{\mathcal{K}_\Theta}M_\Theta = T_\Psi T_\Theta = T_{\Psi\Theta},$$

and hence $\Psi\Theta = 0$. Since Θ is invertible a.e., it follows that $\Psi = 0$ a.e. and hence $T = 0$. \square

Not only is this proposition a considerable generalization of Proposition 2.10 of [5], but our proof is much simpler. The principal tool is the identity $S_\Theta^* = M_z^*|_{\mathcal{K}_\Theta}$.

We have the following characterization which generalizes the characterization of compact operators on \mathcal{K}_Θ for $\dim \mathcal{E} = 1$ (see the implication (i) and (iii) in Theorem 1.2 in [5]).

Theorem 3.5. *Let $\Theta \in H_{\mathcal{B}(\mathbb{C}^p)}^\infty(\mathbb{D})$ be an inner multiplier and $T \in \mathcal{B}(\mathcal{K}_\Theta)$. Assume that $\Theta(e^{i\theta})$ is invertible a.e. Then T is compact if and only if $\{S_\Theta^{*m}TS_\Theta^m\}_{m \geq 1}$ converges in norm.*

Proof. If T is compact on \mathcal{K}_Θ , then by Lemma 3.1, $\|S_\Theta^{*m}TS_\Theta^m\| \rightarrow 0$ as $m \rightarrow \infty$. To prove the converse, let $A \in \mathcal{B}(\mathcal{K}_\Theta)$ and $S_\Theta^{*m}TS_\Theta^m \rightarrow A$ in norm. Then by the same argument used in the proof of Theorem 3.3, we have $S_\Theta^*AS_\Theta = A$. It now follows from Proposition 3.4 that $A = 0$.

Now let R_m , Q_m and F_m , $m \geq 1$, be defined as in the proof of Theorem 3.3. For each $m \geq 1$, define

$$R_{\Theta,m} = S_\Theta^m S_\Theta^{*m},$$

and

$$Q_{\Theta,m} = I_{\mathcal{K}_\Theta} - S_\Theta^m S_\Theta^{*m}.$$

Since

$$S_\Theta = P_{\mathcal{K}_\Theta}M_z|_{\mathcal{K}_\Theta} = P_{\mathcal{K}_\Theta}M_z, \quad \text{and} \quad S_\Theta^* = M_z^*|_{\mathcal{K}_\Theta},$$

for each $m \geq 1$, it follows that

$$R_{\Theta,m} = P_{\mathcal{K}_\Theta}R_m|_{\mathcal{K}_\Theta},$$

and hence

$$Q_{\Theta,m} = P_{\mathcal{K}_\Theta}Q_m|_{\mathcal{K}_\Theta}.$$

Consequently

$$F_{\Theta,m} := TQ_{\Theta,m} + Q_{\Theta,m}T - Q_{\Theta,m}TQ_{\Theta,m} \quad (m \geq 1),$$

is a finite rank operator. Now since

$$T - F_{\Theta,m} = R_{\Theta,m}TR_{\Theta,m} \quad (m \geq 1),$$

we have that

$$\|T - F_{\Theta,m}\| = \|R_{\Theta,m}TR_{\Theta,m}\| = \|S_\Theta^m S_\Theta^{*m}TS_\Theta^m S_\Theta^{*m}\| \leq \|S_\Theta^{*m}TS_\Theta^m\| \rightarrow 0,$$

as $m \rightarrow \infty$. Hence T is compact as required. This completes the proof. \square

Theorem 3.5 and Lemma 3.1 give us the following generalization of Theorem 1.2 in [5].

Theorem 3.6. *Let $\Theta \in H_{\mathcal{B}(\mathbb{C}^p)}^\infty(\mathbb{D})$ be an inner multiplier and $T \in \mathcal{B}(\mathcal{K}_\Theta)$. Then the following are equivalent:*

- (i) $\{S_\Theta^{*m}TS_\Theta^m\}_{m \geq 1}$ converges in norm;
- (ii) $S_\Theta^{*m}TS_\Theta^m \rightarrow 0$ in norm;
- (iii) T is a compact operator.

While our method of proof here is shorter, involves some technical simplifications and is perhaps more transparent, the essential key ideas are taken from [5] and [10].

4. TOEPLITZ OPERATORS IN SEVERAL VARIABLES

In the following we prove a generalization of Brown and Halmos characterization [4] of Toeplitz operators on $H^2(\mathbb{D})$.

Theorem 4.1. *Let $T \in \mathcal{B}(H^2(\mathbb{D}^n))$. Then T is a Toeplitz operator if and only if $M_{z_j}^*TM_{z_j} = T$ for all $j = 1, \dots, n$.*

Proof. For each $k \in \mathbb{N}$, define $\mathbf{k}_d \in \mathbb{N}^n$ by $\mathbf{k}_d = (k, \dots, k)$. From $M_{z_j}^*TM_{z_j} = T$, $j = 1, \dots, n$, we obtain that

$$M_{\mathbf{z}}^{*\mathbf{k}_d}TM_{\mathbf{z}}^{\mathbf{k}_d} = T \quad (k \in \mathbb{N}),$$

which implies that

$$\langle Te_{\mathbf{i}+\mathbf{k}_d}, e_{\mathbf{j}+\mathbf{k}_d} \rangle = \langle TM_{\mathbf{z}}^{\mathbf{k}_d}e_{\mathbf{i}}, M_{\mathbf{z}}^{\mathbf{k}_d}e_{\mathbf{j}} \rangle = \langle Te_{\mathbf{i}}, e_{\mathbf{j}} \rangle \quad (\mathbf{i}, \mathbf{j} \in \mathbb{N}^n).$$

Now for each $\mathbf{l}, \mathbf{m} \in \mathbb{Z}^n$, there exists $\mathbf{t} = (t_1, \dots, t_n) \in \mathbb{N}^n$ such that $\mathbf{l} + \mathbf{k}_d, \mathbf{m} + \mathbf{k}_d \in \mathbb{N}^n$ for all $\mathbf{k}_d \geq \mathbf{t}$ (that is, $k \geq t_j$ for all $j = 1, \dots, n$). Hence setting

$$A_k = M_{e_{i\theta}}^{*\mathbf{k}_d}TP_{H^2(\mathbb{D}^n)}M_{e_{i\theta}}^{\mathbf{k}_d} \quad (k \geq 1),$$

we have

$$\langle A_k e_{\mathbf{l}}, e_{\mathbf{m}} \rangle_{L^2(\mathbb{T}^n)} = \langle TP_{H^2(\mathbb{D}^n)}M_{e_{i\theta}}^{\mathbf{k}_d}e_{\mathbf{l}}, M_{e_{i\theta}}^{\mathbf{k}_d}e_{\mathbf{m}} \rangle_{L^2(\mathbb{T}^n)} = \langle TP_{H^2(\mathbb{D}^n)}e_{\mathbf{l}+\mathbf{k}_d}, e_{\mathbf{m}+\mathbf{k}_d} \rangle_{L^2(\mathbb{T}^n)},$$

and therefore, for all $\mathbf{k}_d \geq \mathbf{t}$, we have that

$$\langle A_k e_{\mathbf{l}}, e_{\mathbf{m}} \rangle_{L^2(\mathbb{T}^n)} = \langle Te_{\mathbf{l}+\mathbf{k}_d}, e_{\mathbf{m}+\mathbf{k}_d} \rangle_{H^2(\mathbb{D}^n)} = \langle Te_{\mathbf{l}+\mathbf{t}}, e_{\mathbf{m}+\mathbf{t}} \rangle_{H^2(\mathbb{D}^n)}.$$

This implies in particular that

$$\langle A_k e_{\mathbf{l}}, e_{\mathbf{m}} \rangle \rightarrow \langle Te_{\mathbf{l}+\mathbf{t}}, e_{\mathbf{m}+\mathbf{t}} \rangle \text{ as } k \rightarrow \infty.$$

Let the bilinear form η on $\text{span}\{e_{\mathbf{s}} : \mathbf{s} \in \mathbb{Z}^n\}$ be defined by

$$\eta(e_{\mathbf{l}}, e_{\mathbf{m}}) = \lim_{k \rightarrow \infty} \langle A_k e_{\mathbf{l}}, e_{\mathbf{m}} \rangle \quad (\mathbf{l}, \mathbf{m} \in \mathbb{Z}^n).$$

Since $\|A_k\| \leq \|T\|$, $k \geq 1$, it follows that η is a bounded bilinear form. Therefore, η can be extended to a bounded bilinear form (again denoted by η) on all of $L^2(\mathbb{T}^n)$, and hence there exists a unique bounded linear operator A_∞ on $L^2(\mathbb{T}^n)$ such that

$$\eta(f, g) = \langle A_\infty f, g \rangle = \lim_{k \rightarrow \infty} \langle A_k f, g \rangle \quad (f, g \in L^2(\mathbb{T}^n)).$$

Now let $j \in \{1, \dots, n\}$, $\mathbf{l}, \mathbf{m} \in \mathbb{Z}^n$ and set $\epsilon_j = (0, \dots, \underbrace{1}_{j^{th} \text{ place}}, \dots, 0)$. Then for all k sufficiently large (depending on \mathbf{l}, \mathbf{m} and j), we have

$$\begin{aligned} \langle (M_{e^{i\theta}}^{*\mathbf{k}_d} T P_{H^2(\mathbb{D}^n)} M_{e^{i\theta}}^{\mathbf{k}_d}) e_{\mathbf{l}+\epsilon_j}, e_{\mathbf{m}+\epsilon_j} \rangle_{L^2(\mathbb{T}^n)} &= \langle T P_{H^2(\mathbb{D}^n)} e_{\mathbf{l}+\mathbf{k}_d+\epsilon_j}, e_{\mathbf{m}+\mathbf{k}_d+\epsilon_j} \rangle_{L^2(\mathbb{T}^n)} \\ &= \langle T e_{\mathbf{l}+\mathbf{k}_d+\epsilon_j}, e_{\mathbf{m}+\mathbf{k}_d+\epsilon_j} \rangle_{H^2(\mathbb{D}^n)} \\ &= \langle M_{z_j}^* T M_{z_j} e_{\mathbf{l}+\mathbf{k}_d}, e_{\mathbf{m}+\mathbf{k}_d} \rangle_{H^2(\mathbb{D}^n)} \\ &= \langle T e_{\mathbf{l}+\mathbf{k}_d}, e_{\mathbf{m}+\mathbf{k}_d} \rangle_{H^2(\mathbb{D}^n)} \\ &= \langle A_k e_{\mathbf{l}}, e_{\mathbf{m}} \rangle_{L^2(\mathbb{T}^n)}. \end{aligned}$$

Therefore

$$\langle A_{\infty} e_{\mathbf{l}+\epsilon_j}, e_{\mathbf{m}+\epsilon_j} \rangle_{L^2(\mathbb{T}^n)} = \lim_{k \rightarrow \infty} \langle (M_{e^{i\theta}}^{*\mathbf{k}_d} T P_{H^2(\mathbb{D}^n)} M_{e^{i\theta}}^{\mathbf{k}_d}) e_{\mathbf{l}+\epsilon_j}, e_{\mathbf{m}+\epsilon_j} \rangle_{L^2(\mathbb{T}^n)} = \langle A_{\infty} e_{\mathbf{l}}, e_{\mathbf{m}} \rangle_{L^2(\mathbb{T}^n)},$$

and consequently $M_{e^{i\theta_j}}^* A_{\infty} M_{e^{i\theta_j}} = A_{\infty}$, that is, $A_{\infty} M_{e^{i\theta_j}} = M_{e^{i\theta_j}} A_{\infty}$. Hence there exists φ in $L^{\infty}(\mathbb{T}^n)$ such that $A_{\infty} = M_{\varphi}$. Finally, we note that for $f, g \in H^2(\mathbb{D}^n)$,

$$\langle A_{\infty} f, g \rangle_{L^2(\mathbb{T}^n)} = \lim_{k \rightarrow \infty} \langle M_{e^{i\theta}}^{*\mathbf{k}_d} T P_{H^2(\mathbb{D}^n)} M_{e^{i\theta}}^{\mathbf{k}_d} f, g \rangle_{L^2(\mathbb{T}^n)} = \lim_{k \rightarrow \infty} \langle M_{\mathbf{z}}^{*\mathbf{k}_d} T M_{\mathbf{z}}^{\mathbf{k}_d} f, g \rangle_{H^2(\mathbb{D}^n)},$$

that is,

$$\langle A_{\infty} f, g \rangle_{L^2(\mathbb{T}^n)} = \langle T f, g \rangle_{H^2(\mathbb{D}^n)},$$

and hence

$$\langle P_{H^2(\mathbb{D}^n)} A_{\infty} f, g \rangle_{H^2(\mathbb{D}^n)} = \langle A_{\infty} f, g \rangle_{L^2(\mathbb{T}^n)} = \langle T f, g \rangle_{H^2(\mathbb{D}^n)}.$$

Therefore, $T = P_{H^2(\mathbb{D}^n)} A_{\infty}|_{H^2(\mathbb{D}^n)} = P_{H^2(\mathbb{D}^n)} M_{\varphi}|_{H^2(\mathbb{D}^n)}$, that is, T is a Toeplitz operator.

Conversely, let $\varphi \in L^{\infty}(\mathbb{T}^n)$ and $T = P_{H^2(\mathbb{D}^n)} M_{\varphi}|_{H^2(\mathbb{D}^n)}$. Then for $f, g \in H^2(\mathbb{D}^n)$ and $j = 1, \dots, n$, we have

$$\langle (M_{z_j}^* T M_{z_j}) f, g \rangle_{H^2(\mathbb{D}^n)} = \langle \varphi e^{i\theta_j} f, e^{i\theta_j} g \rangle_{L^2(\mathbb{T}^n)} = \langle \varphi f, g \rangle_{L^2(\mathbb{T}^n)},$$

that is,

$$\langle (M_{z_j}^* T M_{z_j}) f, g \rangle_{H^2(\mathbb{D}^n)} = \langle P_{H^2(\mathbb{D}^n)} M_{\varphi}|_{H^2(\mathbb{D}^n)} f, g \rangle_{H^2(\mathbb{D}^n)},$$

and therefore $M_{z_j}^* T M_{z_j} = T$ for all $j = 1, \dots, n$, as desired. \square

Theorem 4.1 should be compared with the algebraic characterization of Guo and Wang (see Proposition 2.1 in [14]) which states that T in $\mathcal{B}(H^2(\mathbb{D}^n))$ is a Toeplitz operator if and only if $T_{\varphi}^* T T_{\varphi} = T$ for all inner function $\varphi \in H^{\infty}(\mathbb{D}^n)$.

We now characterize compact operators on $H^2(\mathbb{D}^n)$ in terms of the multiplication operators $\{M_{z_1}, \dots, M_{z_n}\}$. This characterization was proved by Feintuch [10] in the case of $n = 1$.

Theorem 4.2. *A bounded linear map T on $H^2(\mathbb{D}^n)$ is compact if and only if $M_{z_i}^{*m} T M_{z_j}^m \rightarrow 0$ in norm for all $i, j \in \{1, \dots, n\}$.*

Proof. Let T on $H^2(\mathbb{D}^n)$ be a bounded operator. First observe that for each $m \geq 1$, we have

$$M_z^m M_z^{*m} = I_{H^2(\mathbb{D})} - P_{\mathcal{F}_m},$$

where $\mathcal{F}_m = \mathbb{C} \oplus z\mathbb{C} \oplus \cdots \oplus z^{m-1}\mathbb{C}$ is an m -dimensional subspace of $H^2(\mathbb{D})$. Note that for each $m \geq 1$, we have

$$\begin{aligned} F_m &:= \prod_{i=1}^n (I_{H^2(\mathbb{D}^n)} - M_{z_i}^m M_{z_i}^{*m}) \\ &= \prod_{i=1}^n (I_{H^2(\mathbb{D})} \otimes \cdots \otimes (I_{H^2(\mathbb{D})} - M_z^m M_z^{*m}) \otimes \cdots \otimes I_{H^2(\mathbb{D})}) \\ &= \prod_{i=1}^n (I_{H^2(\mathbb{D})} \otimes \cdots \otimes \underbrace{P_{\mathcal{F}_m}}_{i^{th} \text{ place}} \otimes \cdots \otimes I_{H^2(\mathbb{D})}) \\ &= P_{\mathcal{F}_m} \otimes \cdots \otimes P_{\mathcal{F}_m}. \end{aligned}$$

Then F_m is a finite rank operator and hence

$$\tilde{F}_m := TF_m + F_m T - F_m T F_m,$$

is a finite rank operator, $m \geq 1$. Moreover

$$\begin{aligned} T - \tilde{F}_m &= T - (TF_m + F_m T - F_m T F_m) \\ &= (I_{H^2(\mathbb{D}^n)} - F_m)T(I_{H^2(\mathbb{D}^n)} - F_m). \end{aligned}$$

Finally, observe that

$$\begin{aligned} I_{H^2(\mathbb{D}^n)} - F_m &= \sum_{1 \leq i_1 < \cdots < i_l \leq n} (-1)^{l+1} M_{z_{i_1}}^m \cdots M_{z_{i_l}}^m M_{z_{i_1}}^{*m} \cdots M_{z_{i_l}}^{*m} \\ &= \sum_{1 \leq i_1 < \cdots < i_l \leq n} (-1)^{l+1} (M_{z_{i_1}} \cdots M_{z_{i_l}})^m (M_{z_{i_1}} \cdots M_{z_{i_l}})^{*m}, \end{aligned}$$

for all $m \geq 1$. Hence, by hypothesis and the triangle inequality we have

$$\|T - \tilde{F}_m\| = \|(I_{H^2(\mathbb{D}^n)} - F_m)T(I_{H^2(\mathbb{D}^n)} - F_m)\| \rightarrow 0,$$

as $m \rightarrow \infty$. Therefore T is a compact operator.

The converse follows from Lemma 3.1. This completes the proof. \square

Following Feintuch [10] (and Barriá and Halmos [2]) one can now define asymptotic Toeplitz operator as follows:

Definition 4.3. A bounded linear operator T on $H^2(\mathbb{D}^n)$ is said to be an asymptotic Toeplitz operator if there exists $A \in \mathcal{B}(H^2(\mathbb{D}^n))$ such that $M_{z_i}^{*m} T M_{z_i}^m \rightarrow A$ and $M_{z_i}^{*m} (T - A) M_{z_j}^m \rightarrow 0$ as $m \rightarrow \infty$ in norm, $1 \leq i, j \leq n$.

We close this section by characterizing asymptotic Toeplitz operators on $H^2(\mathbb{D}^n)$ as analogous characterization of asymptotic Toeplitz operators on $H^2(\mathbb{D})$ (see Feintuch [10] and Theorem 3.3).

Theorem 4.4. Let T be a bounded linear operator on $H^2(\mathbb{D}^n)$. Then T is an asymptotic Toeplitz operator if and only if T is a compact perturbation of Toeplitz operator.

Proof. Let $A \in \mathcal{B}(H^2(\mathbb{D}^n))$, $M_{z_i}^{*m} T M_{z_i}^m \rightarrow A$ and $M_{z_i}^{*m} (T - A) M_{z_i}^m \rightarrow 0$ in norm, $1 \leq i, j \leq n$. Then for all $m \geq 1$,

$$\begin{aligned} \|A - M_{z_j}^* A M_{z_j}\| &\leq \|A - M_{z_j}^{*(m+1)} T M_{z_j}^{m+1}\| + \|M_{z_j}^{*(m+1)} T M_{z_j}^{m+1} - M_{z_j}^* A M_{z_j}\| \\ &\leq \|A - M_{z_j}^{*(m+1)} T M_{z_j}^{m+1}\| + \|M_{z_j}^{*m} T M_{z_j}^m - A\|, \end{aligned}$$

yields $M_{z_j}^* A M_{z_j} = A$ for all $j = 1, \dots, n$. Also by Theorem 4.2, $T - A$ is compact on $H^2(\mathbb{D}^n)$. The converse follows from Lemma 3.1 and Theorem 4.1. This completes the proof. \square

5. QUOTIENT SPACES OF $H^2(\mathbb{D}^n)$

The purpose of this section is to extend the results of Sections 3 and 4 in the case when the ambient operator is the compression of $(M_{z_1}, \dots, M_{z_n})$ to a joint $(M_{z_1}^*, \dots, M_{z_n}^*)$ -invariant subspace. The proofs in this section will be rather sketchy since they are very similar to the proofs of the corresponding statements in Sections 3 and 4.

Let \mathcal{Q} be a joint $(M_{z_1}^*, \dots, M_{z_n}^*)$ -invariant subspace of $H^2(\mathbb{D}^n)$ and $C_{z_i} = P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}}$, $i = 1, \dots, n$. In this setting, we have the following result:

Theorem 5.1. *Let $T, A \in \mathcal{B}(\mathcal{Q})$, $C_{z_i}^{*m} T C_{z_i}^m \rightarrow A$ and $C_{z_i}^{*m} (T - A) C_{z_i}^m \rightarrow 0$ in norm for all $i, j = 1, \dots, n$. Then $T = A + K$, where $K \in \mathcal{B}(\mathcal{Q})$ is a compact operator and $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \dots, n$.*

Proof. The proof is similar to that of Theorems 3.5, 4.2 and 4.4, and therefore is omitted. \square

Considering the particular case $\mathcal{Q} = H^2(\mathbb{D}^n)/\Theta H^2(\mathbb{D}^n)$, where $\Theta \in H^\infty(\mathbb{D}^n)$ is an inner function, we get the following result.

Proposition 5.2. *Let $\Theta \in H^\infty(\mathbb{D}^n)$ be an inner function and $\mathcal{Q} = H^2(\mathbb{D}^n)/\Theta H^2(\mathbb{D}^n)$ and $A \in \mathcal{B}(\mathcal{Q})$. Then $C_{z_i}^* A C_{z_i} = A$ for all $i = 1, \dots, n$, if and only if $A = 0$.*

Proof. The proof goes exactly along the same lines as the proof of Proposition 3.4. Since

$$A P_{\mathcal{Q}} = (C_{z_i}^* A C_{z_i}) P_{\mathcal{Q}} = M_{z_i}^* (A P_{\mathcal{Q}}) M_{z_i} \quad (i = 1, \dots, n),$$

it follows from Theorem 4.1 that $A P_{\mathcal{Q}}$ is a Toeplitz operator. Again, since M_{Θ} is an analytic Toeplitz operator, it follows that $A P_{\mathcal{Q}} M_{\Theta} = 0$. Hence, using [Theorem 1, C. Gu [13]], we conclude that $A = 0$. \square

Summing up the above two results and Lemma 3.1, we have the following generalization of Theorem 1.2 in [5].

Theorem 5.3. *For an inner function $\Theta \in H^\infty(\mathbb{D}^n)$ and bounded linear operators T and A on $\mathcal{Q} = H^2(\mathbb{D}^n)/\Theta H^2(\mathbb{D}^n)$, the following are equivalent:*

- (i) $C_{z_i}^{*m} T C_{z_i}^m \rightarrow A$ and $C_{z_i}^{*m} (T - A) C_{z_i}^m \rightarrow 0$ in norm for all $i, j = 1, \dots, n$;
- (ii) $C_{z_i}^{*m} T C_{z_i}^m \rightarrow 0$ in norm for all $i = 1, \dots, n$;
- (iii) T is compact.

For asymptotic Toeplitzness of composition operators on the Hardy space of the unit sphere in \mathbb{C}^n we refer the reader to Nazarov and Shapiro [18], and Cuckovic and Le [7].

Acknowledgement: The first author's research work is supported by NBHM Post Doctoral Fellowship No. 2/40(50)/2015/ R & D - II/11569. The second author is supported in part by NBHM (National Board of Higher Mathematics, India) grant NBHM/R.P.64/2014.

REFERENCES

- [1] P. Ahern, E. Youssfi and K. Zhu, Compactness of Hankel operators on Hardy-Sobolev spaces of the polydisk. *J. Operator Theory* 61 (2009), 301-312.
- [2] J. Barría and P. R. Halmos, Asymptotic Toeplitz operators. *Trans. Amer. Math. Soc.* 273(2):621-630, 1982.
- [3] A. Bottcher and B. Silbermann, *Analysis of Toeplitz operators*. Springer-Verlag, Berlin, 1990.
- [4] A. Brown and P. R. Halmos, Algebraic properties of Toeplitz operators. *J. Reine Angew. Math.* 213:89-102, 1963/1964.
- [5] I. Chalendar, W. T. Ross, Compact operators on model spaces, 2016.(arXiv:1603.01370).
- [6] B. Choe, H. Koo and Y. Lee, Commuting Toeplitz operators on the polydisk. *Trans. Amer. Math. Soc.* 356 (2004), 1727-1749.
- [7] Z. Cuckovic and T. Le, Toeplitzness of composition operators in several variables. *Complex Var. Elliptic Equ.* 59 (2014), 1351-1362.
- [8] X. Ding, Products of Toeplitz operators on the polydisk. *Integral Equations Operator Theory.* 45 (2003), 389-403.
- [9] R. G. Douglas, *Banach algebra techniques in operator theory*. Second edition. Graduate Texts in Mathematics, 179. Springer-Verlag, New York, 1998.
- [10] A. Feintuch, On asymptotic Toeplitz and Hankel operators. In *The Gohberg anniversary collection*, Vol. II (Calgary, AB, 1988), volume 41 of *Oper. Theory Adv. Appl.*, pages 241-254. Birkhauser, Basel, 1989.
- [11] S. Garcia, J. Mashreghi, and W. Ross, *Introduction to model spaces and their operators*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2016.
- [12] G. Popescu, Free pluriharmonic functions on noncommutative polyballs. *Anal. PDE* 9 (2016), 1185-1234.
- [13] C. Gu, Some algebraic properties of Toeplitz and Hankel operators on polydisk. *Arch. Math. (Basel)* 80 (2003), 393-405.
- [14] K. Guo and K. Wang, On operators which commute with analytic Toeplitz operators modulo the finite rank operators. *Proc. Amer. Math. Soc.* 134 (2006), 2571-2576.
- [15] P. Hartman and A. Wintner, The spectra of Toeplitz's matrices. *Amer. J. Math.* 76 (1954), 867-882.
- [16] S. Sun and D. Zheng, Toeplitz operators on the polydisk. *Proc. Amer. Math. Soc.* 124 (1996), 3351-3356.
- [17] B. Sz.-Nagy and C. Foias, *Harmonic Analysis of Operators on Hilbert Space*. North-Holland, Amsterdam-London, 1970.
- [18] F. Nazarov and H. Shapiro, On the Toeplitzness of composition operators. *Complex Var. Elliptic Equ.* 52 (2007), 193-210.
- [19] W. Rudin, *Function theory in polydiscs*. W. A. Benjamin, Inc., New York-Amsterdam 1969.
- [20] H. Upmeyer, *Toeplitz operators and index theory in several complex variables*. Operator Theory: Advances and Applications, 81. Birkhauser Verlag, Basel, 1996.

INDIAN STATISTICAL INSTITUTE, STATISTICS AND MATHEMATICS UNIT, 8TH MILE, MYSORE ROAD,
BANGALORE, 560059, INDIA

E-mail address: amaji_pd@isibang.ac.in, amit.iitm07@gmail.com

INDIAN STATISTICAL INSTITUTE, STATISTICS AND MATHEMATICS UNIT, 8TH MILE, MYSORE ROAD,
BANGALORE, 560059, INDIA

E-mail address: jay@isibang.ac.in, jaydeb@gmail.com

INDIAN STATISTICAL INSTITUTE, STATISTICS AND MATHEMATICS UNIT, 8TH MILE, MYSORE ROAD,
BANGALORE, 560059, INDIA

E-mail address: srijan_rs@isibang.ac.in, srijansarkar@gmail.com